Section 5.1: The Natural Logarithmic Function: Differentiation

In the previous chapter, it was noted that the antiderivative of a power function is given by

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

However, this operation is not defined when $n = 1$, because it results in a division by zero. Looking at the graph of 1/x, one can see that it should have a definite integral for any $x > 0$. We *define* the integral of $1/t$ with respect to t over the interval $[1, x]$ to be ln x where ln x is the known as the *natural logarithm* function.

$$\int_1^x \frac{1}{t} dt = \ln x$$

Applying the fundamental theorem of calculus, we see that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Or, more generally,

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

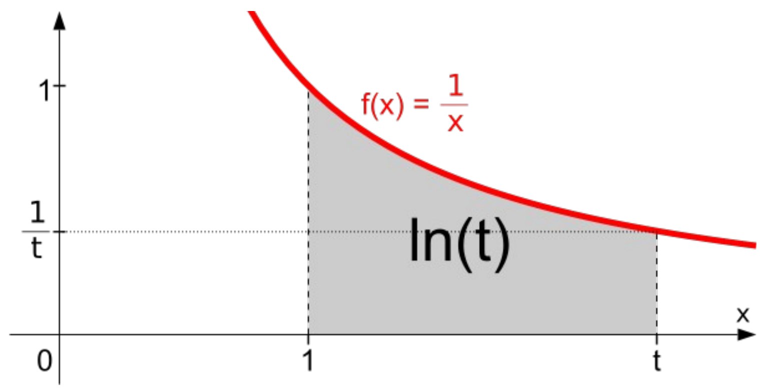


Figure 1: The area under the graph of the function $f(x) = 1/x$ on the interval $[1, t]$ is the equal to ln t

Using this latter equation, the familiar properties of ln x can be easily derived:

$$\ln ab = \ln a + \ln b$$
$$\ln \frac{a}{b} = \ln a - \ln b$$
$$\ln x^n = n \ln x$$

By substituting $x^n = 2^{2N}$ where N is an arbitrarily large number into the last of these equations, we see that

$$\ln 2^{2N} = 2N \ln 2 > N$$

Therefore, the natural logarithm of a number can be made arbitrarily large.

$$\lim_{x \to \infty} \ln x = \infty$$

In fact, of all integrals of the form, $\lim_{x \to \infty} \int_1^x t^n dt$, $n = -1$ is the smallest value of n for which the integral is infinite.

Logarithmic differentiation

When differentiating functions involving natural logarithms, it is often expedient to rewrite the function using the laws of logarithms. For instance,

$$\frac{d}{dx} = \ln \frac{x^3\sqrt{x+3}}{(x^2-3)^2} = \frac{d}{dx}\left(3\ln x + \frac{1}{2}\ln(x+3) - 2\ln(x^2-3)\right) = \frac{3}{x} + \frac{1}{2(x+3)} - \frac{4x}{x^2-3}$$

In some cases it may be advantageous to approach a difficult derivative problem by taking the natural logarithm of both sides and differentiating implicitly rather than by attacking the derivative directly. For instance, *dy*/*dx* may be calculated as follows:

$$\begin{aligned} y &= \frac{(x-2)^2}{\sqrt{x^2+1}} \\ \ln y &= \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \\ &= 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \\ \frac{y'}{y} &= \frac{2}{x-2} - \frac{x}{x^2+1} \\ y' &= y\left[\frac{2}{x-2} - \frac{x}{x^2+1}\right] \\ &= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{2}{x-2} - \frac{x}{x^2+1}\right] \end{aligned}$$

This process is known as *logarithmic differentiation.*

Section 5.2: The Natural Logarithmic Function: Integration

In the last section it was noted that the integral

$$\int_1^x \frac{1}{t}\,dt = \ln x$$

Or more generally, for $x \in [a, b]$

$$\int_a^b \frac{1}{t}\,dt = \ln b - \ln a$$

But this was defined only for x values > 0. Because $y = 1/x$ is an odd function, the integral from $-b$ to $-a$ should be the same but negative:

$$\int_{-b}^{-a} \frac{1}{t}\,dt = \ln a - \ln b$$

We may combine the last two equations by stating that the general antiderivative of the function $y = 1/x$ is

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Moreover,

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

The latter formula, in combination with substitution and other integration techniques allows us to integrate a large number of functions. In particular, any function that can be written all or partly as a derivative divided by its parent function will involve a logarithm. In particular, the integral of the tangent function may be written as

$$\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$$

A similar technique applies to the other trig functions as well. We can now summarizes the antiderivatives of the six basic trig functions as shown below:

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

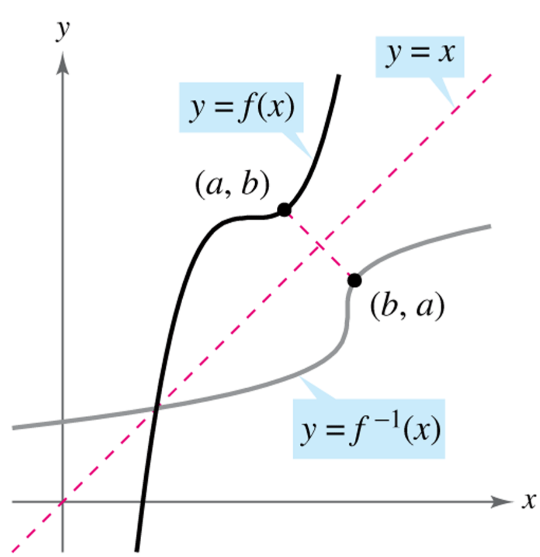
Section 5.3: Inverse Functions

If $f(x)$ is a function, then the *inverse function* $f^{-1}(x)$ corresponding to $f(x)$ is the function that has the properties

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(x)) = x$$

where the domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.

In general, $f^{-1}(x)$ can be thought of as the function that *undoes* $f(x)$. For instance, if $f(x) = x + 3$, then $f^{-1}(x) = x - 3$.



The equation of $f^{-1}(x)$ may be found by replacing y in the definition of $y = f(x)$ with x and x with y and solving for y algebraically.

Since (y, x) is the point on the $f^{-1}(x)$ corresponding to (x, y) on $f(x)$, $f(x)$ and $f^{-1}(x)$ are reflectionally symmetrical about the line $y = x$. Moreover, the slope $\Delta y/\Delta x$ at a point on $f(x)$ is the reciprocal of the corresponding point on $f^{-1}(x)$. In terms of derivatives, if $g(x)$ is an inverse function of $f(x)$, then

$$\boxed{g'(x) = \frac{1}{f'(g(x))}}$$

For example, the slope of $f(x)$ is 3 at the point (2, 7), then the slope of $f^{-1}(x)$ at the point (7, 2) is ⅓.

This is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

although it is important to note that in general these derivatives are not evaluated at the same points.